On Learning Processes and the National Mathematics Advisory Panel Report

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This article is a response to *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (2008) and to one of the task group reports on which it was based, the report of the Task Group on Learning Processes. The author uses Maxwell's two views of causality—regularity and process—to explore three major issues raised in the report: the nature of what is learned, how learning occurs, and the transfer of learning. She proposes alternative recommendations to those offered by the Panel, by drawing upon the mathematics education literature, which was largely excluded from the reviewed research. Furthermore, by neglecting research grounded in a process view of causality, the report excludes perspectives and findings that would have illuminated what it means to develop conceptual understanding in mathematics—one of three valued outcomes cited by the Panel.

Keywords: learning processes and strategies; mathematics education; policy; transfer

During a recent trip to China, I was surprised to learn of the keen interest of Chinese researchers in the final report of the National Mathematics Advisory Panel (NMAP; 2008). Such interest speaks to the potential global influence of the report and to the significance of the critiques found in this special issue of *Educational Researcher.* In this article, I explore the following issues raised in chapter 5 of the report ("Learning Processes"): (a) the nature of what is learned, (b) how learning occurs, and (c) the application or transfer of learning.

I propose alternative interpretations and recommendations to those offered in the report by drawing upon the mathematics education literature, which was largely excluded from the reviewed research. As indicated in the Task Group on Learning Processes report (Geary et al., 2008), all major mathematics education journals were excluded (including the *Journal for Research in Mathematics Education,* the *Journal of Mathematical Behavior, Educational Studies in Mathematics,* and *Mathematical Thinking and Learning*). The journals deemed acceptable rely heavily on experimental psychology (e.g., *Journal of Experimental Psychology*).

Because the NMAP report draws primarily on journals from experimental and educational psychology, it is important to recognize general differences in the goals and assumptions of psychologists and mathematics educators. According to DeCorte, Greer, and Verschaffel (1996), experimental psychologists tend to treat mathematics as the topic that is codified in commonly used textbooks. In contrast, mathematics educators often provide new conceptual analyses of content and create instructional designs to support "what might be" rather than "what is" (Bishop, 1992).

In a related point, Cobb (2007) argues that experimental psychology focuses on how variations in instructional conditions affect the learning outcomes for a statistically constructed collective subject. In contrast, mathematics education research tends to focus on how students come to reason in qualitatively different ways by successively reorganizing their activity. Stated more broadly, Maxwell (2004) would say that experimental psychologists tend to hold a *regularity* view of causality. From this perspective, causality cannot be directly observed but can only be measured in the regularity of relationships between events. This necessitates the determination of a systematic relationship between inputs (e.g., an experimental condition or a curricular treatment) and outputs (e.g., a subject's performance in an experiment or the learning outcomes for a group of students). Mathematics education researchers typically rely on a different but compatible aspect of scientific explanation, which Maxwell calls *process causality.* This is based on a conceptual analysis of processes by which some events influence others. Whereas regularity causation is most useful in establishing that something happens, process causality addresses why or how it happened.

These distinctions are important to a discussion of learning processes. By restricting the criteria for inclusion to those research studies that utilized "experimental, quasi-experimental, or correlational methods" (Geary et al., 2008, p. 153), the task group limited its ability to respond to its stated goal to "address what is known about *how* [italics added] children learn and understand areas of mathematics related to algebra and preparation for algebra" (p. xi).

The Nature of What Is Learned

The learning processes chapter of the final report includes the following statement, which suggests a welcome broadening of the types of mathematical knowledge that are valued:

> Debates regarding the relative importance of conceptual knowledge, procedural skills (e.g., the standard algorithms), and the commitment of addition, subtraction, multiplication, and division facts to long-term memory are misguided. These capabilities are mutually supportive, each facilitating learning of the others. Conceptual understanding of mathematical operations, fluent execution of

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> procedures, and fast access to number combinations together support effective and efficient problem solving. (NMAP, 2008, p. 26)

However, the NMAP report does not operationalize the meanings of *conceptual understanding* and *problem solving* with definitions or illustrative examples. Because each of these terms has multiple meanings, it is possible for groups with different values and priorities to use the same language but talk past each other (Skemp, 1976). For example, a teacher may say that a student understands the concept of division of fractions, but mean that the student can efficiently perform the invert-and-multiply algorithm. Here *concept* is used to mean topic and *understanding* to mean computational fluency. Alternatively, a teacher may say that a student understands the concept of triangle, and mean that the child can accurately classify simple shapes, in which case *concept* means category. In a third characterization, conceptual understanding "refers to an integrated and functional grasp of mathematical ideas" (Kilpatrick, Swafford, & Findell, 2001, p. 118). This includes having meaning for mathematical symbols and operations, interpreting results, making connections among ideas or representations, and understanding why particular procedures work. To illustrate the types of statements that would have been useful to include in the report, three concepts related to division of fractions are articulated below:

Concept 1. One way to interpret the meaning of division by a fraction is as the number of the divisor that fits in the dividend, which is referred to as the measurement interpretation of division. For example, $8 \div {}^{1}/_{4}$ is the number of one-fourths of 1 whole that fit in 8 wholes.

Concept 2. In a measurement interpretation of division by a fraction, the remainder is interpreted in terms of the divisor rather than in terms of the units associated with the dividend. For example, if one interprets $2 \div {}^{3}/_{5}$ as the number of ${}^{3}/_{5}$ in 2 wholes, then there is one ${}^{3}/_{5}$ in each of 2 wholes. A third ${}^{3}/_{5}$ can be formed by gathering the leftover fifths. This still leaves ${}^{1}/_{5}$ of 1 whole. Rather than writing the quotient as $3^{1}/_{5}$, the ${}^{1}/_{5}$ needs to be interpreted in terms of the divisor. Because ${}^{1}/_{5}$ is ${}^{1}/_{3}$ of ${}^{3}/_{5}$, then $2 \div {}^{3}/_{5} = 3^{1}/_{3}$.

Concept 3. The invert-and-multiply procedure for dividing by a fraction works for the following the reason. Inverting the divisor yields its reciprocal, which can be interpreted as the number of the divisor that fits in 1 whole. Because the dividend represents the total number of wholes and because the goal is to determine the number of the divisor that fits in the dividend, the reciprocal is then multiplied by the dividend. Reconsider the previous example of $2 \div {}^{3}/_{5}$. This time, we start by considering a simpler problem, namely, to find the number of ${}^{3}/_{5}$ in 1 whole (instead of 2 wholes). Using Concept 2, we can reason that there are $1^{2}/_{3}$ three-fifths in 1 whole, which is the same as saying that there are ${}^{5}/_{3}$ three-fifths in 1 whole. Note that ${}^{5}/_{3}$ is reciprocal of ${}^{3}/_{5}$. Returning to the original problem of locating the number of ${}^{3}/_{5}$ in 2 wholes, we simply multiply the number of ${}^{3}/_{5}$ in 1 whole (${}^{5}/_{3}$) by the number of wholes (2). Thus, to determine $2 \div {}^{3}/_{5}$, we inverted and multiplied: ${}^{5}/_{3} \times 2$.

Without such elaborations of conceptual knowledge, it is easy to end up focusing almost exclusively on the types of knowledge for which shared meaning exists, namely, procedural knowledge and basic arithmetic facts. For example, the *Mathematics Framework for California Public Schools* (California Department of Education, 2000) states that students must become "proficient in basic computation and procedural skills, develop conceptual understanding, and become adept at problem solving. All three components are important; none is to be neglected or underemphasized" (p. 7). However, the content standards in the same document focus almost exclusively on procedures, facts, and definitions. Similarly, the NMAP report identifies grade-level benchmarks to serve as critical foundations for algebra. Yet these are stated in terms of procedural fluency; for example, "By the end of Grade 6, students should be proficient with multiplication and division of fractions and decimals" (p. 20). The report (2008) also identifies Major Topics of School Algebra (see p. 16), which consists of a list of topics (e.g., logarithmic functions, complex numbers, and rational expressions) and procedures (e.g., factoring quadratic polynomials with integer coefficients).

The NMAP report would have made a significant contribution to the future of mathematics education had it specified the conceptual understandings that should be achieved in algebra and pre-algebra topics. Although documents such as the *Principles and Standards for School Mathematics* (PSSM; National Council of Teachers of Mathematics, 2000) broaden state standards, the articulation of concepts in the PSSM is neither complete nor sufficiently elaborated. For example, the PSSM identifies the measurement meaning of division by a fraction, but the other concepts noted previously (such as the interpretation of remainders and why the algorithm works) are not articulated. At a minimum, the NMAP report should have illustrated the meaning of the term *conceptual understanding* and provided a rationale for its importance.

Elaborating a set of concepts is a difficult task, even for (maybe especially for) mathematically sophisticated adults. This is the case in part because what is difficult for a child to understand early on in the development of an idea is often no longer apparent to an adult looking through the lens of sophisticated understanding (Simon, 2006). As a result, qualitative research, especially conducting clinical interviews with children, is useful to the process of articulating conceptual knowledge. One way to identify key understandings is to observe students engaged in mathematical tasks and then to specify understandings that can account for variation in the actions of students or differences in the ways children reason compared to adults.

It is especially important to investigate how children reason when the tasks evoke their everyday experiences. For example, in an interview study with high school students, we asked them to create a way to measure the steepness of a wheelchair ramp (Lobato, 2008b; Lobato & Thanheiser, 2002). More than half of the students struggled to isolate the attribute of steepness from other attributes, such as the work required to climb the ramp. Many students talked about the importance of including the length of the slanted part of the ramp in their measures of steepness because a longer ramp is more difficult to climb (i.e., a person is slowed down as he or she moves up the ramp). To better understand this difficulty, consider two nonidentical ramps with the same steepness (Figure 1). Many students will draw on their

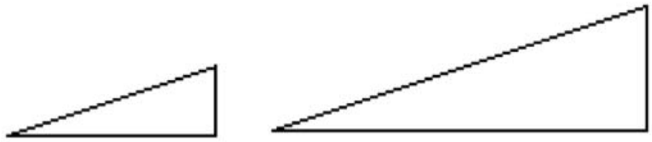

FIGURE 1. *Two nonidentical ramps with the same steepness.*

everyday experiences of walking on ramps to argue that the ramp on the right is steeper because it is higher, longer, or harder to climb. If students do not isolate the attribute of steepness from the attribute of the work required to climb the ramp, then they may conclude incorrectly that the ramps are not the same steepness because people will become more tired when climbing the ramp on the right. By reflecting on these results, we developed a conceptual analysis of the mathematical topic of slope, which included the isolation of the attribute to be measured by slope (here, the steepness of the ramp) from other attributes in the situation (see Lobato, 2008b, for details).

Such conceptual analyses informed by children's reasoning exist for many topics in mathematics but appear largely in mathematics education journals or in well-respected edited books, which were excluded from the NMAP report. Consequently, the Panel would have done the field a service had it recommended that a synthesis of existing research be conducted to compile known key conceptual understandings into a coherent, well-sequenced list for heavily researched areas and to recommend that additional research be conducted in under-researched topics (such as quadratic functions, trigonometric relationships, complex numbers, etc.).

Part of the NMAP statement regarding conceptual knowledge, procedural skills, basic facts, and problem solving is that these capabilities are "mutually supportive, each facilitating learning of the others." This is a plausible statement, which refers to a complex set of relationships. However, the report does not (a) specify how one type of knowledge is used to create the other type for each knowledge pair and in each direction, (b) identify a curricular order based on either a theoretical argument or empirical evidence, or (c) articulate the types of pedagogical moves that help children create such connections. Lacking these critically important elaborations permits superficial interpretations of the statement, such as teaching part of the time for meaning and part of the time for recall and procedural skill development.

For example, if a sixth-grade teacher introduces the invert-and-multiply rule for dividing fractions on one day and the measurement meaning of division of fractions the next day, no one would expect this to result in students having an understanding of the invert-and-multiply rule as expressed in Concept 3 above. In fact, exploratory empirical research in mathematics classrooms by Pesek and Kirshner (2000) suggests that teaching for procedural skill development prior to teaching for conceptual understanding can interfere with meaning development. Much more research is needed to understand the complex set of relationships that exists between conceptual knowledge, procedural skills, basic facts, and problem solving.

How Learning Occurs

Although no learning processes are articulated in the final report, several mechanisms from an information-processing perspective (including attention, encoding, and working memory) are reviewed in the task group report. The only other theoretical perspective on learning acknowledged by the NMAP report is the sociocultural perspective of Vygotsky, which is then promptly dismissed as follows: "A shortage of controlled experiments makes the usefulness of Vygotsky's approach for improving mathematics learning difficult to evaluate, and thus its utility in mathematics classrooms and mathematics curricula needs to be scientifically tested" (Geary et al., 2008, p. xiii).

This dismissal of a major theoretical perspective and a large body of associated research in mathematics education is puzzling until one considers Maxwell's (2004) distinction between regularity causality and process causality. In the task group report, learning is characterized as "causal pathways to successful outcomes" (Geary et al., 2008, p. 2), and learning processes are conceived as "cognitive factors . . . that have been linked to achievement outcomes" (p. 3). According to Maxwell, the regularity view treats causality as unobservable and focuses instead on a systematic relationship between inputs and outputs (here, the factors that are linked with increased performance on learning outcomes).

In contrast, in the process view, the search is for a causal mechanism rather than for causal effect. Whereas the regularity view holds that causality can never be identified in single events, the process view allows the possibility of inferring cause in single cases. Researchers operating from a Vygotskian approach are interested in causal explanations of what brings about advances in children's participation in social practices. In particular, they are concerned with what brings about a reduction in the zone of proximal development, which is "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). Researchers then look for conceptual connections between contingent interactions and children's advances in order to build causal explanations of these advances (the positing of learning processes), often by carefully analyzing learning events occurring in a single situation, without comparison to a control group (see Roth, 2008, for a more elaborated response to the NMAP report regarding sociocultural learning theories).

To dismiss Vygotskian theory as a resource for the identification of learning process on the grounds that controlled experiments were not conducted indicates a serious lack of understanding of the need for different types of evidence to establish causal explanation versus causal effect. This point is illustrated by the task group's inclusion of a quotation taken out of context. After reviewing the van Hiele model of geometric reasoning, the task group states that it agrees with Battista (2007) with respect to the more difficult task of identifying the cognitive processes underlying the van Hiele levels:

> Although a number of theories and studies have been reviewed in an attempt to describe the cognitive processes by which students progress through the early van Hiele levels, this area of research is still in its infancy. . . . To achieve progress in this domain, it is important for mathematics education researchers to heed the work of researchers in other fields such as cognitive science and neuroscience. Such research can provide valuable insights into these

> difficult-to-observe processes. (Battista, 2007, pp. 858–859; cited by Geary et al., 2008, p. 71)

However, the task group neglected to include the rest of the quote from Battista:

> But we must also continue to conduct studies that expose students to instructional treatments and tasks and carefully investigate phenomenologically how students construct meaning for geometric concepts, always attempting to build cognitive models that explain what we observe. It is through the integration of cognitive and phenomenological investigations that we can make true progress in this vital area of research. (Battista, 2007, p. 859)

This passage indicates an equal valuing of studies that examine advances in how children construct meaning in ecologically valid contexts such as classrooms, innovative teaching experiments, and everyday situations (not just in laboratories).

The NMAP report's dismissal of a Vygotskian approach becomes more serious when one realizes that other major theoretical perspectives important to mathematics education research were also overlooked. These include Piagetian constructivism (von Glasersfeld, 1995), situated cognition (Brown, Collins, & Duguid, 1989), and the emergent perspective (Cobb & Yackel, 1996). Studies guided by these perspectives have also been grounded largely in a process view of causality.

Research from these perspectives has identified at least three types of learning processes. First, general processes that hold across disciplines have been identified, including (a) accommodation and reflective abstraction from a Piagetian constructivist perspective (Steffe, 1991; von Glasersfeld, 1991); (b) internalization, semiotic meditation, and appropriation from a Vygotskian perspective (John-Steiner & Mahn, 1996); (c) social norms and the negotiation of mathematical meaning from an emergent perspective (Cobb & Yackel, 1996); and (d) legitimate peripheral participation from a situated cognition perspective (Lave & Wenger, 1990).

Second, domain-specific theories have been developed that are intended to offer insights into the learning of any mathematical topic. For example, Pirie and Kieren's (1994) recursive theory of mathematical understanding identifies eight levels of mathematical reasoning and accounts for understanding as a recursive phenomenon that occurs as thinking moves between levels of sophistication. According to Cobb (2007), learning processes have also been identified that cut across various math topics but do not comprise as expansive a theory as the Pirie and Kieran model. Examples include the processes of reification (Sfard & Linchevski, 1994), unitizing (Steffe, 1992), norming (Lamon, 1994), and splitting (Confrey, 1994). (One of these processes is elaborated later in the article.)

Finally, learning trajectories have been created for particular topics in mathematics. These trajectories typically identify cognitive milestones and the processes by which children use prior knowledge to construct more sophisticated understanding. For example, Battista and colleagues developed a learning trajectory for area and volume measurement that articulates seven levels of sophistication in students' structuring and enumeration of arrays, as well as five underlying mental processes (Battista, 2004, 2007; Battista & Clements, 1996; Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998).

To illustrate what would have been gained by including these learning processes in the NMAP report, I compare several learning processes identified in the task group report related to the learning of fractions, with two learning processes identified in the omitted mathematics education literature. The task group report reviews several studies grounded in a regularity view of causality, which investigated the effects of various factors—working memory capacity, paying attention to instruction, and basic arithmetic skills—on outcome measures, including the ability to solve fraction computation, estimation, and word problems. The task group concluded that fraction instruction should begin by emphasizing quick retrieval of basic arithmetic facts because committing such facts to memory reduces working memory demands and allows attention to be focused on other problem features.

What is missing from this account is an articulation of the particular cognitive processes by which children form fraction meaning from their understandings of whole numbers. Two such processes—*unitizing* and *partitioning*—have been identified from studies grounded in a process view of causality in which children's reasoning was analyzed as they developed fraction understanding (Olive, 1999; Tzur, 1999, 2000). Lamon (1994) provides an overview of the critical role of unitizing in both whole number and rational number development. For example, when a child forms a *composite unit* of whole numbers (a type of unitizing), say a "five," he or she mentally bundles 5 ones into a single unit (a 5-unit), which, in turn, can be unpacked into 5 ones. To see the role of forming composite units in fraction development, consider the meaning of the fraction ¾ in the context of finding ¾ of 20 marbles. This example also involves a second fundamental learning process, namely that of *partitioning* or forming equal-sized groups. The 20 marbles can first be partitioned into four groups of equal size. Children often accomplish this by dealing the 20 marbles across four imagined people or by splitting the group of 20 in half and in half again. Understanding that one of the resulting equal groups of 5 marbles also represents ¼ of 20 involves the coordination of units of units. The 20 single marbles can be thought of as a single composed unit (a 20-unit), which in turn, is composed of 4 units of 5-units. The 5-unit can then be considered as ¼ of the total number of marbles, and ¾ of 20 can be considered as a set of 3 units of the 5-unit, which is 15 marbles. Identification of these learning processes suggests instructional recommendations such as providing children with varied opportunities to partition sets of objects and to engage in the types of reflective practices in classrooms that help them coordinate various levels of units.

The Transfer of Learning

Because the Panel adopted an information-processing perspective on learning, it is not surprising that the report relies exclusively on a traditional view of transfer. However, numerous critiques of the traditional view of transfer (summarized in Lobato, 2006) have contributed to a growing acknowledgment that "there is little agreement in the scholarly community about the nature of transfer, the extent to which it occurs, and the nature of its underlying mechanisms" (Barnett & Ceci, 2002, p. 612). Several alternative transfer approaches have emerged in response to the critiques: (a) the *preparation for future learning* approach (Bransford & Schwartz, 1999; Schwartz & Martin, 2004), (b) the *actor-oriented transfer* perspective (Lobato, 2003, 2006, 2008b), (c) transfer as *consequential transitions* (Beach, 1999, 2003), and (d) the *affordances and constraints* approach (Greeno, Smith, & Moore, 1993).

The importance of these alternative perspectives on transfer is witnessed by two conferences on transfer funded by the National Science Foundation (see Lobato, 2004; Mestre, 2003), several recent books (Mestre, 2005; Tuomi-Gröhn & Engeström, 2003), and the 3-year transfer strand sponsored by the *Journal of the Learning Sciences.* A viable conjecture regarding why the NMAP report omitted a discussion of alternative perspectives could be that these perspectives do not rely on the methods valued by the Panel. However, the preparation for future learning perspective not only utilizes experimental methods but has been published in journals deemed "acceptable" by the NMAP report (e.g., Schwartz & Martin, 2004); yet it was also omitted from inclusion in the task group report. My goal in this section is to briefly contrast the traditional transfer model with one of these alternative perspectives to demonstrate how the interpretation of research findings and corresponding instructional recommendations change under an alternative perspective on transfer.

Transfer is defined in the NMAP report (2008) as "the ability to correctly apply one's learning beyond the exact examples studied" (p. 7). This characterization is consistent with the *common elements* theories that have dominated the 20th century, starting with Thorndike's (1906) emphasis on identical elements in the physical environment to information-processing accounts of identical or overlapping mental schemes (e.g., Anderson, Corbett, Koedinger, & Pelletier, 1995). In a typical study of transfer, researchers present participants with a sequence of tasks that share some structural features (e.g., a common solution approach or shared principle) but have different surface forms (e.g., different word problem contexts), according to the researcher's (or other expert's) knowledge of the topic. Participants are then taught some solution strategy, principle, or procedure with the initial learning task. If the participants perform better on a transfer task than a control group does (the control group receives the transfer task but no learning tasks), then transfer is said to have occurred.

Traditional transfer studies privilege the perspective of the observer and rely on models of expert performance (Evans, 1998; Lobato, 2003). As a result, transfer experiments can become what Lave (1988) calls an "unnatural, laboratory game in which the task becomes to get the subject to match the experimenter's expectations," rather than an investigation of the "processes employed as people naturally bring their knowledge to bear on novel problems" (p. 20). In contrast, the actor-oriented transfer perspective (Lobato, 2003) seeks to understand the ways in which people generalize their learning experiences rather than predetermining what counts as transfer, using models of expert performance. From this perspective, psychological similarity—how a new situation is connected with a person's experience of a previous situation—rather than similar elements of task features serves as the basis of transfer. From the actor-oriented perspective, *transfer* is defined as the generalization of learning, which can also be understood as the influence of learners' prior activities on their activities in novel situations (Lobato, 2003, 2008a).

Adopting an alternative view of transfer has consequences for the interpretation of research and for instructional recommendations. Consider the following recommendation offered by the NMAP report (2008):

> Conceptual and procedural knowledge about fractions with magnitudes less than 1 do not necessarily transfer to fractions with magnitudes greater than 1. Therefore, understanding of fractions with magnitudes in each range needs to be taught directly, and the relation between them needs to be discussed. (p. 28)

To explore this statement, consider a child who is shown a square divided into 4 equal parts with 3 parts shaded. When asked to write the fraction of the shape that is shaded, the child counts the number of parts that are shaded (3) and the total number of parts (4), states that "3 out of the 4 parts are shaded," and correctly writes $^3/_4$. However, when asked to draw a figure to show the improper fraction $^5/_4$, the child responds incorrectly that "it can't be done because you can't have 5 of something when there are only 4 parts." From a traditional transfer perspective, there was no transfer of the child's previous fraction knowledge. From an actor-oriented perspective, there was complete transfer of the "parts of" conception of fractions.

Mathematics education research points to the "iterative fraction scheme" as a more powerful conception of fractions (Steffe, 1993; Tzur, 1999). For example, here one thinks of $^3/_4$ as 3 one-fourths, where one-fourth is conceived as the result of partitioning a whole into 4 equal parts. As a result, it is possible to have any number of one-fourths. For example, $^5/_4$ is interpreted as 5 one-fourths, which can also be conceived as 4 one-fourths or 1 whole plus an additional one-fourth; $^8/_4$ is interpreted as 8 one-fourths, which can be grouped into two sets of 4 one-fourths, or 2 wholes, and so on. From an actor-oriented transfer perspective, the appropriate instructional recommendation is *not* to teach proper and improper fractions separately, but rather to help children construct the generalizable iterative fraction scheme in the initial learning situation, as opposed to unwittingly supporting the construction of the "parts of" conception of fractions.

In a related statement regarding the frequency by which transfer occurs, the final report states, "Studies of transfer suggest that people's ability to make links between related domains is limited" (NMAP, 2008, p. 30). Indeed, traditional transfer studies often fail to demonstrate transfer in the laboratory, which has led some to conclude that transfer is rare (Detterman, 1993). Others have interpreted the poor results as evidence that reasoning is hopelessly context bound or "that our cognitive apparatus simply does not incline very much to transfer" (Perkins & Salomon, 1989, p. 22). However, the traditional transfer approach underestimates the generalization of learning by accepting as evidence of transfer only specific correspondences defined a priori as the "right" mappings (Lobato, 2008a). It is not surprising then that we find overwhelming evidence for the lack of transfer from the traditional perspective, because we know that novices do not make the same set of connections as experts do (National Research Council, 2000). In contrast, use of the actor-oriented transfer approach allows for instances of generalizing that would not be counted as transfer in the traditional approach and helps guard against false conclusions regarding the degree to which humans generalize (Lobato, 2008b; Lobato & Siebert, 2002). In sum, the NMAP would have done the research community a service if it had acknowledged the challenges to the traditional perspective on transfer, reviewed the literature on alternative transfer perspectives, and recognized that new approaches lead to equally valid interpretations of research results with important implications for instruction.

Concluding Remarks

When I asked the Chinese researchers why they were so interested in the NMAP report, they responded that they see the United States as a leader in innovation and they expected the report to be "forward looking." Indeed, the report could have contributed significantly to an innovative vision had it specified the conceptual understandings that should be achieved in algebra and related pre-algebra topics. Without sufficient elaboration of the meanings of *conceptual understanding* and *problem solving*—terms that lack shared meaning in our culture—the report emphasized the types of knowledge for which there is shared meaning: procedural knowledge and basic arithmetic facts. Innovation would have been achieved by an acknowledgment of the need for causal explanation as well as for causal effect—two compatible aspects of scientific explanation. A more forward-looking synthesis of learning processes exists in the *Second Handbook of Research on Mathematics Teaching and Learning* (Lester, 2007). Finally, innovation would have been achieved with the acknowledgment that the traditional view of transfer has been critiqued since the 1980s, that alternative perspectives on transfer have emerged as a result, and that different implications for instruction arise from these new perspectives.

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