On Professional Judgment and the National Mathematics Advisory Panel Report: Curricular Content

Patrick W. Thompson

In *Foundations for Success: The Final Report of the National Mathematics Advisory Panel* (2008), the Panel's recommendations on curricular content are based on the professional judgment of the Panel members, not on research on learning and teaching algebra. As such, the Panel's recommendations must be viewed in light of the political and ideological perspectives from which they were made. The recommendations are also examined from the perspective that inattention to meaning is a root problem of mathematics teaching and learning, yet the report does not consider issues of mathematical meaning or their implications for reform.

Keywords: coherence; curriculum; learning; mathematics; meaning

President George W. Bush directed the National Mathematics Advisory Panel (NMAP) "to use the best available scientific evidence" to devise a report addressing, among other things, "instructional practices, programs, and materials that are effective for improving mathematics learning" and "ideas for strengthening capabilities to teach children and youth basic mathematics, geometry, algebra, and calculus and other mathematical disciplines" (NMAP, 2008a, p. 7). As I argue below, the Panel's response falls short of the president's charge in two ways. First, the report fails to produce a trustworthy argument for the curricular recommendations it gives. Second, the recommendations themselves address only surface aspects of the problems in American mathematics education.

Trustworthiness

The Panel defined "best available scientific evidence" to mean results gotten from experimental studies that test hypotheses, that use random selection and randomized assignment to treatments, and that have been replicated (NMAP, 2008a, p. 81). Although the Panel gave two standards of quality—one for evidence and one for effects of interventions—they are both made from an experimental design perspective. It is instructive that the Panel's quality-of-evidence scheme has a "suggestive evidence" category, but to be suggestive a study must still use statistical controls. Moreover, to be above "low quality" as an investigation of effect, a study must have at least a "moderately large" probability sample (pp. 83–85). So, by the Panel's definitions, most qualitative studies, especially teaching experiments and design experiments, fall under the Panel's low-quality category and cannot provide even suggestive evidence for future research. As a result, the report is not informed by a large portion of basic mathematics education research that investigates fundamental processes of classroom learning and teaching. This last point is expanded in a later section.

The report conforms for the most part to its standards when it does cite a source of evidence when making an evidence-based claim or recommendation. Unfortunately, the Panel does not refrain from taking stances for which it has no empirical evidence that meets its standard. In particular, the report of the Task Group on Conceptual Knowledge and Skills (NMAP, 2008b), which developed chapter 4 ("Curricular Content"), stated, "It should be noted that there is no direct empirical evidence to support the effectiveness of any lists discussed in this section for success in algebra course work" (p. 31). The task group nevertheless went on to construct recommendations that were "guided by professional judgment" (pp. ix, x, 2, 5, 15, 31, 40). However, the Panel's final report was less forthcoming in regard to the task group report's reliance on professional judgment, stating,

> A small number of questions have been deemed to have such currency as to require comment from the Panel, even if the scientific evidence was not sufficient to justify research-based findings. In those instances, the Panel has spoken on the basis of collective professional judgment, but it has also endeavored to minimize both the number and the scope of such comments. (NMAP, 2008a, p. 12)

Despite the Panel's claim of having minimized both the number and scope of comments based on "collective professional judgment," the curricular content group's recommendations, all of which were arrived at by professional judgment, are included in chapter 4 of the Panel's report, and there are no other recommendations in that chapter. However, the phrase "professional judgment" appears just once in chapter 4.

It is important to understand the significance of the Panel's deep reliance on the task group's professional judgment in formulating its curricular recommendations. First, doing so belied the Panel's own standards of quality and evidence and its claim

Educational Researcher, Vol. 37, No. 9, pp. 582–587
DOI: 10.3102/0013189X08328000

that it would make evidence-based recommendations. Second, although the Panel chose to rely on some of its members' professional judgment in formulating its recommendations, it consciously chose to ignore the professional judgment of more reform-minded experts who had perspectives that differed from the Panel's.

Moreover, the task group's heavy reliance on professional judgment to make its recommendations calls into question its members' qualifications for exercising such judgment. For example, what biases did they have? How familiar were they with, and how well did they understand, the basic research literature on mathematics teaching and learning? The Panel's deep reliance on professional judgment in formulating its curricular content recommendations, without a transparent effort to represent competing viewpoints, leaves the Panel open to the charge that it might be composed of the most politically connected members of the community but not the most knowledgeable regarding mathematics learning, teaching, and curricula.

Chapter 4 is written to convey a solid scholarly grounding when often there is none. For example, the claim that "fractions, when properly taught, introduces students to the use of symbolic notation and the concept of generality" (NMAP, 2008b, p. 41) seems innocuous. However, in the task group's report, the warrant for this claim comes from one reference to one expository article by one of the Panel's members (Wu, 2001). The reference is omitted from the Panel's report, however, giving the statement the appearance that it is a straightforward generalization from literature that the Panel found to support it. Now, one might say that literature does exist that suggests that fractions, properly taught, is a site where students can learn to reason generally, but the Panel did not find it. Rather, it expressed its opinion and hid the fact that it was expressing ungrounded opinion. That is, the Panel's violation of its own evidence standards in formulating its recommendations creates the impression that the Panel picked its evidence to suit its biases.

The filter that the task group applied to the research literature on mathematics learning, teaching, and curricula produces an ironic result. As a result of focusing only on investigations of efficacy or effect that passed through its self-defined evidence and quality standards, the task group does not cite a single result from basic research in mathematics education. At the same time, it found that studies of effect and efficacy of instructional or curricular approaches that meet its standards are rare. The net effect, then, is that chapter 4 of the Panel's report is neither informed by basic research in mathematics education nor grounded in high-quality studies of efficacy and effect. Put another way, the Panel's content recommendations neither convey a reason for believing that these recommendations will work (because they are uninformed by any mechanisms from basic research by which they might be expected to work) nor inspire confidence that they will be effective (because of the paucity of experimental comparisons).

Had the Panel adopted a more inclusive attitude toward the literature it could examine, it would have found a plethora of important issues it should address—in students' preparation for algebra and in the very nature of algebra for which they should be prepared. Regarding students' preparation for algebra, it is well known that many students do not make a transition from arithmetic to algebra. The literature on epistemological and cognitive obstacles to mathematical learning makes clear that the way a student understands a particular idea can enable or obstruct the student's learning of ideas that depend on it (Booth, 1981; Carlson, 1998; Harel, 1998; Herscovics, 1989; Herscovics & Linchevski, 1994; Schneider, 1991; Sierpinska, 1992). For example, students in U.S. elementary grades rarely write arithmetic sentences to represent a situation, whereas Russian, Chinese, and Japanese students do (Fuson, Stigler, & Bartsch, 1989). The rarity of this practice in U.S. elementary grades has the effect that when students move from arithmetic to algebra they have developed a predisposition to understand expressions as something they should calculate rather than to understand them as representations to interpret (Herscovics & Linchevski, 1994; Kieran, 1992; Linchevski & Herscovics, 1996). Regarding the nature of algebra, it is well known that a strong focus on algebra as symbol manipulation has the effect that students rarely develop the idea that variables represent quantities whose values vary, which has long-term negative effects for students' success in calculus and analytic geometry (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Oehrtman, Carlson, & Thompson, 2008; Schneider, 1991, 1992; Thompson, 1994a; Trigueros & Jacobs, 2008).

Finally, it is important to note that the Panel's report has a decidedly martial tone. By that I mean that it focuses on a need for students to exhibit specific correct mathematical behavior. The tension within the Panel over this issue is reflected in its discussion of whether to refer to algorithms, standard algorithms, or "the" standard algorithms (November 28, 2007, meeting transcript, pp. 115–120, 168–170). The "the" faction won—the report always takes it as a primary goal that knowing an algorithm that produces correct results is not satisfactory. Rather, students must learn specific ("the standard") algorithms in arithmetic.

In this same regard, there is a stark contrast between the tone of the 2008 Japanese Course of Study (COS; Japan Ministry of Education, 2008) and the Panel report's very martial tone. The COS is quite concerned with children's thinking; the Panel report and the task group report are more concerned with children's behavior. A comparison of the Panel and task group reports with the Japanese COS also highlights the Panel's inattention to the role of meaning in learning mathematics. By "meaning," I do *not* mean "definition." Rather, I use "meaning" in the sense of that which comes to mind to make a word, phrase, observation, and so on, sensible and comprehensible. Anyone familiar with the Japanese K–9 curriculum knows the importance it gives to students' construction of meaning as the basis for skill. The comparison also highlights an internal contradiction in the Panel's report—it calls repeatedly for greater coherence in the mathematics curriculum while seemingly being unaware that coherence is a property of a body of meanings, not a property of a list of topics. Thus, the task group report provides no guidance to the nation about how to increase the likelihood that mathematics instruction and curricula will end up with students creating coherent mathematical meanings during their schooling.

Recommendations for Curricular Content

With the above said about the weak warrant given for the recommendations in chapter 4 and about the strong skills-without-meaning perspective taken by the Panel, there is little to say about the substance of the chapter's content recommendations. They revolve around a list of 27 "major topics of school algebra" and a list of 11 "benchmarks" for what the Panel considers "critical foundations" for school algebra. Unfortunately, the list of major topics is as informative about what the Panel envisions being taught as is a textbook's table of contents. Likewise, the benchmarks are little more than statements that students will be proficient with the procedures that are taught at a particular grade level. Anyone looking for guidance on actually implementing the Panel's vision of preparing students for algebra will not find it in this report.

One would hope that the task report provides insights not included in the Panel report. It does not. The one area that showed promise but remained unfulfilled was the task group's emphasis on the importance that students develop strong facility with fractions. The Panel stated the following recommendation:

> The curriculum should allow for sufficient time to ensure acquisition of conceptual and procedural knowledge of fractions (including decimals and percents) and of proportional reasoning. The curriculum should include representational supports that have been shown to be effective, such as number line representations, and should encompass instruction in tasks that tap the full gamut of conceptual and procedural knowledge, including ordering fractions on a number line, judging equivalence and relative magnitudes of fractions with unlike numerators and denominators, and solving problems involving ratios and proportion. The curriculum also should make explicit connections between intuitive understanding and formal problem solving involving fractions. (p. 29)

Few people in mathematics education would disagree with this recommendation. But preceding it is a sequence of paragraphs talking about the necessity that children be able to "quickly and easily retrieve basic number facts" (NMAP, 2008a, p. 28), which sets an unfortunate tone for interpreting the recommendation, and statements about how little is understood about the relationship between "informal knowledge and the learning of formal mathematical fractional concepts and procedures" (p. 28). This latter statement is not true. The complexities of this relationship are well understood and have been explicated thoroughly (Ball, 1993; Cramer, Post, & delMas, 2002; Empson, Junk, Dominguez, & Turner, 2006; Heller, Post, Behr, & Lesh, 1990; Mack, 2001; Norton, 2008; Sowder, Bezuk, & Sowder, 1993; Steffe, 2001; Thompson & Saldanha, 2003; Tirosh, 2000). But this understanding did not show up in the literature that the Panel surveyed. The Panel would have served the nation well had it explained why this understanding has not affected the treatment of fractions in school texts nor the general psychological literature on fractions.

Competing Visions of School Algebra and Preparation for Them

It should be clear that there is little chance that the Panel's report will lead to improvements in students' learning of arithmetic or algebra. Its emphasis on proficiency with standard procedures in arithmetic and its lip service to "conceptual understanding" will do little to address the fundamental problem of mathematics education in the United States—namely, the systematic inattention to students' development of meanings that will support an interest in mathematics that results in taking more, and higher level, coursework.

A variety of sources suggest that incoherence and meaninglessness is a prominent feature of school mathematics in the United States. The Third International Mathematics and Science Study (TIMSS) report of 8th-grade mathematics instruction in the United States, Germany, and Japan states this clearly:

> Finally, as part of the video study, an independent group of U.S. college mathematics teachers evaluated the quality of mathematical content in a sample of the video lessons. They based their judgments on a detailed written description of the content that was altered for each lesson to disguise the country of origin (deleting, for example, references to currency). They completed a number of in-depth analyses, the simplest of which involved making global judgments of the quality of each lesson's content on a three-point scale (Low, Medium, High). Quality was judged according to several criteria, including the coherence of the mathematical concepts across different parts of the lesson, and the degree to which deductive reasoning was included. Whereas 39 percent of the Japanese lessons and 28 percent of the German ones received the highest rating, none of the U.S. lessons received the highest rating. Eighty-nine percent of U.S. lessons received the lowest rating, compared with 11 percent of Japanese lessons [and 34% of the German lessons]. (Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999, p. iv)

The TIMSS video sampling technique drew representative samples from each of its participating countries (Stigler et al., 1999). That no U.S. lesson's content received the highest quality rating from these mathematicians and that 89% of the U.S. lessons' content received the lowest quality rating suggests a general inattention in U.S. instruction to meaning in general, let alone meanings that students develop. Instead, U.S. lessons tended to focus on having students do things and remember what they had done. There was little emphasis on having students develop robust and generalizable meanings. However, lack of meaning in instruction is a problem that cannot be addressed by a revised list of arithmetic or algebra topics. It will require a systemic reorientation toward students' long-term development of usable and powerful meanings and toward an increase in their proclivity to reason with them. To attain a focus on students' development of powerful mathematical meanings and using them in their reasoning, which includes coming to a consensus on what accomplishing it looks like, will require sustained political and intellectual leadership. Unfortunately, the Panel report provides neither one. Whether all students learn the same computational algorithms is a distraction.

The Panel report conveys a stance that meaning is not relevant. How is this so? By, for example, speaking of solving equations as a necessary skill at which students must become proficient, but not raising the importance that students develop meanings for equations—"Where do equations come from? Why are they important? What, precisely, do they represent?"—and develop meanings for equivalent expressions and equivalent equations (Ernest, 1987; Kirshner, 1989). It stresses the importance of linear functions but not the importance of establishing coherent understandings of direct variation and constant rate of change as a foundation for them (Clagett, 1968; Confrey, 1994; Harel, Behr, Lesh, & Post, 1994; Hart, 1978; Karplus, Pulos, & Stage, 1983; Thompson, 1994b). It stresses the importance of proficiency with standard arithmetic algorithms but does not admit the importance of, or the subtleties of, having students develop orientations to "making sense" of problems and situations in which conventional procedures might be used (Cobb et al., 1991). It does not recognize that a single-minded focus on proficiency is likely one of the major sources of the mathematics anxiety that is rampant in our population (Bessant, 1995; DeBellis & Goldin, 2006; Skemp, 1979; Wilensky, 1997). It fails to recognize the importance of what Harel (1998) has called "the necessity principle," the principle that students learn mathematics more meaningfully when ideas are introduced to satisfy an intellectual need for them and that such needs are engineered by thoughtfully crafted curricula and instruction. For example, if the question were instead, "Could we rewrite this expression so that we know it represents the same number, but does more useful work?" then properties such as distributivity could be made *necessary* in a context. The Mathematical Association of America report *Algebra as a Gateway to a Technological Future* (McCallum et al., 2007) does a nice job of making this argument.

Finally, the vision of algebra reflected in the Panel's content recommendations is a skills-based foundation for advanced symbolic manipulation and abstract algebra (especially the algebra of polynomial forms). It completely ignores algebra as a preparation for calculus, which would entail strong emphases on variable as varying magnitude (Trigueros & Jacobs, 2008), covariation and function (Carlson et al., 2002; Oehrtman et al., 2008), rate of change and accumulation (Confrey & Smith, 1995; Thompson & Silverman, 2008), and modeling (Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Lehrer, Schauble, Carpenter, & Penner, 2000; Smith, Haarer, & Confrey, 1997; Verschaffel, De Corte, & Vierstraete, 1999). Even successful university calculus students have difficulty solving problems that depend on understanding ideas such as varying magnitude and function as a relationship between variables (Carlson, 1998; A. Selden, Mason, & Selden, 1994; A. Selden, Selden, Hauk, & Mason, 2000; J. Selden, Mason, & Selden, 1989). All these examples point to the importance that the specific meanings that students create for foundational ideas have significant long-term repercussions in their mathematical development.

Most of the studies cited here are qualitative studies or have small samples that were not selected at random. Nevertheless, as a whole, they suggest that the task group's focus on skills is misplaced. For the task group to use the literature I cited, however, would have required that its members know and value it. With a few exceptions, there is no evidence that they did. Instead, they relied on a Defense Department group to sift their literature for them. It is thus disappointing that the Panel missed a once-in-a-generation opportunity to address foundational problems in the nation's mathematics education. Instead, it gives untrustworthy arguments for its curricular recommendations, and the recommendations themselves address only surface aspects of the problems in American mathematics education.

Finally, I should point out that most of the studies I have cited in criticism of the Panel's report and the recommendations created by the task group are, according to the Panel's standards of evidence and quality, of weak (or no) evidence and low quality. There is no doubt that research in mathematics education needs more experimental studies that test hypotheses and add to theory confirmation. However, the funding has never been adequate both to support research into basic mechanisms of learning and teaching and to support large-scale, randomized experimental studies.

REFERENCES

Ball, D. L. (1993). Halves, pieces, and twoths: Constructing and using representational contexts in teaching fractions. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 157–195). Hillsdale, NJ: Lawrence Erlbaum.

Bessant, K. C. (1995). Factors associated with types of mathematics anxiety in college students. *Journal for Research in Mathematics Education, 26*(4), 327–345.

Booth, L. R. (1981). Child-methods in secondary mathematics. *Educational Studies in Mathematics, 12*(1), 29–41.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In *Research in collegiate mathematics education* (CBMS Issues in Mathematics Education, Vol. 7, pp. 114–162). Washington, DC: Mathematical Association of America.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Clagett, M. (1968). *Nicole Oresme and the medieval geometry of qualities and motions*. Madison: University of Wisconsin Press.

Cobb, P., Wood, T., Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B., et al. (1991). Assessment of a problem-centered second grade mathematics project. *Journal for Research in Mathematics Education, 22*(2), 3–29.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 293–330). Albany, NY: SUNY Press.

Confrey, J., & Smith, E. (1995). Splitting, covariation and their role in the development of exponential function. *Journal for Research in Mathematics Education, 26*(1), 66–86.

Cramer, K. A., Post, T. R., & delMas, R. C. (2002). Initial fraction learning by fourth-and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the rational number project curriculum. *Journal for Research in Mathematics Education, 33*(2), 111–144.

DeBellis, V., & Goldin, G. A. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies in Mathematics, 63*(2), 131–147.

Empson, S. B., Junk, D., Dominguez, H., & Turner, E. (2006). Fractions as the coordination of multiplicatively related quantities: A cross-sectional study of children's thinking. *Educational Studies in Mathematics, 63*(1), 1–28.

Ernest, P. (1987). A model of the cognitive meaning of mathematical expressions. *British Journal of Educational Psychology, 57,* 343–370.

Fuson, K., Stigler, J. W., & Bartsch, K. (1989). Grade placement of addition and subtraction topics in Japan, Mainland China, the Soviet Union, and the United States. *Journal for Research in Mathematics Education, 19*(5), 449–456.

Gravemeijer, K. P. E., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolizing, modeling, and instructional design. In P. Cobb & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 225–274). Hillsdale, NJ: Lawrence Erlbaum.

Harel, G. (1998). Two dual assertions: The first on learning and the second on teaching (or vice versa). *Mathematical Monthly, 105*(6), 497–507.

Harel, G., Behr, M., Lesh, R. A., & Post, T. (1994). Constancy of quantity's quality: The case of the quantity of taste. *Journal for Research in Mathematics Education, 25*(4), 324–345.

Hart, K. (1978). The understanding of ratios in secondary school. *Mathematics in School, 7,* 4–6.

Heller, P. M., Post, T. R., Behr, M., & Lesh, R. A. (1990). Qualitative and numerical reasoning about fractions and rates by seventh- and eighth-grade students. *Journal for Research in Mathematics Education, 21*(5), 388–402.

Herscovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 60–86). Hillsdale, NJ: Lawrence Erlbaum.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27*(1), 59–78.

Japan Ministry of Education. (2008). *Japanese mathematics curriculum in the course of study* (A. Takahashi, T. Watanabe, & Y. Makoto, Trans.). Madison, WI: Global Education Resources.

Karplus, R., Pulos, S., & Stage, E. (1983). Early adolescents' proportional reasoning on "rate" problems. *Educational Studies in Mathematics, 14*(3), 219–234.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan.

Kirshner, D. (1989). The visual syntax of algebra. *Journal for Research in Mathematics Education, 20*(3), 274–287.

Lehrer, R., Schauble, L., Carpenter, S., & Penner, D. E. (2000). The interrelated development of inscriptions and conceptual understanding. In P. Cobb & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design* (pp. 325–360). Hillsdale, NJ: Lawrence Erlbaum.

Linchevski, L., & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. *Educational Studies in Mathematics, 30*(1), 39–65.

Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education, 32*(3), 267–295.

McCallum, W., Thompson, P. W., Harel, G., Blaire, R., Dance, R., Nolan, E., et al. (2007). Intermediate algebra. In V. J. Katz (Ed.), *Algebra: Gateway to a technological future* (pp. 19–27). Washington, DC: Mathematical Association of America.

National Mathematics Advisory Panel. (2008a). *Foundations for success: The final report of the National Mathematics Advisory Panel.* Washington, DC: U.S. Department of Education.

National Mathematics Advisory Panel. (2008b). *Foundations for success: Report of the Task Group on Conceptual Knowledge and Skills.* Washington, DC: U.S. Department of Education.

Norton, A. (2008). Josh's operational conjectures: Abductions of a splitting operation and the construction of new fractional schemes. *Journal for Research in Mathematics Education, 39*(4), 401–430.

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27–42). Washington, DC: Mathematical Association of America.

Schneider, M. (1991). Un obstacle épistémologique soulevé par des "découpages infinis" des surfaces et des solides [An epistemological obstacle raised by "infinite surface divisions" and solids]. *Reserches en Didactique des Mathématiques, 11*(2, 3), 241–294.

Schneider, M. (1992). A way to analyse several difficulties the pupils meet with in calculus. In *Proceedings of the 7th International Congress of Mathematical Education* (Working Group on Students' Difficulties in Calculus, pp. 31–34). Québec: Les Presses de l'Université Laval.

Selden, A., Mason, A., & Selden, J. (1994). Even good calculus students can't solve nonroutine problems. In J. J. Kaput & E. Dubinsky (Eds.), *Research issues in undergraduate mathematics learning: Preliminary analyses and results* (MAA Notes, Vol. 33, pp. 19–26). Washington, DC: Mathematical Association of America.

Selden, A., Selden, J., Hauk, S., & Mason, A. (2000). Why can't calculus students access their knowledge to solve non-routine problems? In J. J. Kaput, A. H. Schoenfeld, & E. Dubinsky (Eds.), *Research in collegiate mathematics education* (Vol. 4, pp. 128–145). Washington, DC: CBMS.

Selden, J., Mason, A., & Selden, A. (1989). Can average calculus students solve non-routine problems? *Journal of Mathematical Behavior, 8*(1), 45–50.

Sierpinska, A. (1992). On understanding the notion of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 25–58). Washington, DC: Mathematical Association of America.

Skemp, R. (1979). *Intelligence, learning, and action.* New York: John Wiley.

Smith, E., Haarer, S., & Confrey, J. (1997). Seeking diversity in mathematics education: Mathematical modeling in the practice of biologists and mathematicians. *Science and Education, 6*(5), 441–472.

Sowder, J. T., Bezuk, N., & Sowder, L. K. (1993). Using principles from cognitive psychology to guide rational number instruction for prospective teachers. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 239–258). Hillsdale, NJ: Lawrence Erlbaum.

Steffe, L. P. (2001). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior, 20*(3), 267–307.

Stigler, J. W., Gonzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (1999). *The TIMSS Videotape Classroom Study: Methods and findings from an exploratory research project on eighth-grade mathematics instruction in Germany, Japan, and the United States* (National Center for Education Statistics Report No. NCES 99–0974). Washington, DC.: U.S. Government Printing Office.

Thompson, P. W. (1994a). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2–3), 229–274.

Thompson, P. W. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179–234). Albany, NY: SUNY Press.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin. & D. Schifter (Eds.), *Research companion to the Principles and Standards for School Mathematics* (pp. 95–114). Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43–52). Washington, DC: Mathematical Association of America.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5–25.

Trigueros, M., & Jacobs, S. (2008). On developing a rich conception of variable. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (MAA Notes, pp. 3–13). Washington, DC: Mathematical Association of America.

Verschaffel, L., De Corte, E., & Vierstraete, H. (1999). Upper elementary school pupils' difficulties in modeling and solving nonstandard additive word problems involving ordinal numbers. *Journal for Research in Mathematics Education, 30*(3), 265–285.

Wilensky, U. (1997). What is normal anyway? Therapy for epistemological anxiety. *Educational Studies in Mathematics, 33*(2), 171–202.

Wu, H.-H. (2001, Summer). How to prepare students for algebra. *American Educator,* 1–7.

AUTHOR

PATRICK W. THOMPSON is a professor of mathematics education in the School of Mathematics and Statistics at Arizona State University, P.O. Box 871804, Tempe, AZ 85287-1804; *pat.thompson@asu.edu*. His research interests are mathematics learning, teaching, and curricula, and secondary mathematics teacher professional development.

Manuscript received July 29, 2008
Revision received October 12, 2008
Accepted October 17, 2008